A Simplified Non-Gaussian Mixture Model for Signal LO Detection in α-Stable Interference

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Abstract

Non-Gaussian statistic model, alpha stable distribution has gained much attention due to its generality to represent heavy-tailed and impulsive interference. Unfortunately, there is no closed form expression for the probability density function of alpha-stable distributions. Hereby, to develop the approximate expressions is of importance for signal detection and denoising. This paper is concerned with the weak signal detection in α-stable interference. We present a novel approximate expression that is a simplified version of Cauchy-Gaussian mixture (CGM) for symmetric α-stable (SαS) distribution. Consequently, the non-linearity limiter of locally optimal (LO) detector is deduced. Compared with Cauchy detector, the proposed detector has near optimal performance.

1. Introduction

Signal detection, which detects the presence of a signal in noisy observations, is a classical problem that has to be implemented in a variety of applications, the more obvious ones being in radar, sonar and communications. The signal detection problems usually are viewed as problems of hypothesis testing in statistical inference [1] in which the generalized likelihood ration test (GLRT) is the most widely accepted method of solution. In most of previous work on detection, it has been assumed that the signal is embedded in Gaussian noise and the detectors are designed accordingly. The Gaussian noise assumption has been generally justified with the central limit theorem and with the analytical convenience of the Gaussian probability density function (PDF) which leads to linear and hence tractable equations [2]. However, there are also many cases in detection, in which the noise is decidedly non-Gaussian. Non-Gaussianity often results in significant performance degradation for detector designed under the Gaussian assumption.

The non-Gaussian noise in practice can be characterized by its impulsive nature. As a result, its density functions in the tail heavier than those of Gaussian density. Since 1991, there has been a tremendous interest in the class of α-stable distributions [3], which are a generalization of Gaussian distribution, but are able to model a wider range of phenomena and can be of a more impulsive nature [4,5,6].

In a detection problem, optimal processing is feasible if the noise PDF is analytically known and tractable. Unfortunately, no closed forms for the probability densities of alpha stable distribution except for three special cases, Gaussian, Cauchy and Pearson distribution [3]. Therefore, to search approximation mixture model is a feasible way. Currently, there are two classes of approximation mixture models, one is scale mixture of the Gaussian, and it is popular to approximate the PDF by a finite Gaussian mixture model (GMM) [2]. Although GMM fits the $S\alpha S$ distribution well, it cannot capture the algebraic tails of alpha stable distributions with small number of Gaussian components *N* and loss analytical convenience with large *N*. The other one is Cauchy Gaussian mixture model (CGM) that is more expensive computationally in evaluation triple parameters [7]. In order to develop tractable approximation model, we introduce a simplified CGM called SCGM with bi-parameter, which fits $S\alpha S$ density well meanwhile keeps less computational burden. Based on such model, the zero-memory non-linear (ZMNL) function is deduced and corresponding adaptive Locally Optimum (LO) detectors is constructed.

This paper is organized as follows. Section 2 explains the main concept of alpha stable distribution parameters α , σ and μ . The SCGM model is proposed in Section 3, The corresponding ZMNL and adaptive LO detector are presented in Section 4. The experimental results for detection of deterministic direct current signal in α-stable interference are shown in the last.

2. Alpha stable distribution

A random variable x is said to have a alpha-stable distribution $S_\alpha(\sigma,\beta,\mu)$ if there are parameters $0<\alpha\le 2$, $-1\le\beta\le 1$, $\sigma>0$, $\mu\in\mathbb{R}$ so that its characteristic function has the following form [3]:

$$\Phi(\theta)=E\exp[j\theta x]$$

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$$
= \begin{cases} \exp\left\{ j\mu\theta - |\sigma\theta|^{\alpha} \left(1 - j\beta sign(\theta) \tan\frac{\pi\alpha}{2}\right)\right\}, & \alpha \neq 1 \\ \exp\left\{ j\mu\theta - |\sigma\theta| \left(1 + j\beta\frac{2}{\pi} sign(\theta) \ln|\theta|\right)\right\}, & \alpha = 1 \end{cases} \tag{1}
$$

where $sign(\theta) = \begin{cases} 1, & \theta > 0 \\ 0, & \theta = 0 \\ -1, & \theta < 0 \end{cases}$.

An alpha stable distribution is completely determined by four parameters: the characteristic exponent α, the index of skewness β, the scale exponent σ and the location parameter μ. The distribution is said to be symmetric alpha stable ($S\alpha S$) when $\beta = 0$ and a stable distribution is called standard if $\mu = 0, \sigma = 1$. If $\alpha \neq 1$, the cases $\beta > 0$ and $\beta < 0$ correspond to positive skewness and negative skewness respectively.

The tails of alpha stable distribution decrease like a power function and the rate of decay depends on characteristic exponent α, as shown in Fig.1, the smaller α, the slower the decay and the heavier the tails.

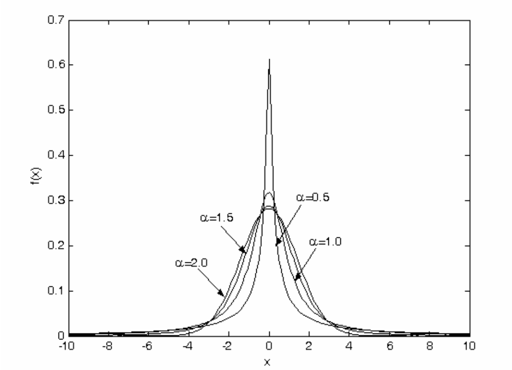

Fig.1 Graphs of standard $S\alpha S$ densities corresponding to the value $\alpha = 0.5$, $\alpha = 1.0$, $\alpha = 1.5$ and $\alpha = 2.0$

There are no closed forms for the probability densities of alpha stable distribution except for three special cases. The case of $\alpha = 2, \beta = 0$ corresponds to the Gaussian distribution, while $\alpha = 1, \beta = 0$ corresponds to the Cauchy distribution and $\alpha = 0.5, \beta = 1$ corresponds to the Pearson distribution. The density functions with $\mu = 0$ in such three cases are given by

$$
f_G(x) = \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{x^2}{4\sigma^2}\right) \tag{2}
$$

$$
f_C(x) = \frac{\sigma}{\pi(x^2 + \sigma^2)} \tag{3}
$$

$$
f_P(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{x^{3/2}} \exp\left\{-\frac{\sigma}{2x}\right\} \tag{4}
$$

3. Non-Gaussian mixture model

In a detection problem, optimal processing is feasible if the noise PDF is analytically known and tractable. Gaussian mixture models are popular because of their universal approximation properties and have been used to model impulsive noise. The classical Middleton Class A models are Poisson weighted Gaussian mixture models where the variances increase linearly. Since the $S\alpha S$ random variable can be represented as a scale mixture of the Gaussian, it is natural to approximate its PDF by a finite Gaussian mixture model (GMM) which is based on the following theorem [2,3].

Theorem 1. Let *X* be distribution with Gaussian distribution, $X \sim N(0, \sigma^2)$. Also let *Y* be s positive stable random variable, $Y \sim S_{\alpha/2}\left(\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha}, 1, 0\right)$ and be independent from *X*, then, $Z = Y^{1/2} X \sim S_{\alpha}(\sigma, 0, 0)$.

Consequently, *Z* is a compound random variable. By letting $V = Y^{1/2}$, one can express that PDF of *Z* in the following way.

$$
f_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v^{-1} \exp\left(-\frac{z^2}{2v^2}\right) f_V(v) dv \tag{5}
$$

The PDFs of Z that can be expressed as in equation (5) are called scale mixtures of normal distributions. In addition, equation (5) can be approximated by finite mixture model with arbitrary parameters

$$
f_Z(z) = \frac{\sum_{i=1}^{N} \frac{1}{\sqrt{2\pi} v_i} \exp\left(-\frac{x^2}{2v_i^2}\right) f_V(v_i)}{\sum_{i=1}^{N} f_V(v_i)} \tag{6}
$$

Kuruoglu proposed using the Expectation Maximization (EM) algorithm to obtain finial approximation with 10 Gaussian terms [2]. To achieve accurate approximation, the number of Gaussian components *N* usually is larger than eight [8]. Although GMM fits the $S\alpha S$ distribution well, it cannot capture the algebraic tails of alpha stable distributions with small *N* and loss analytical convenience with large *N*.

The other one class of non-Gaussian mixture model is Cauchy Gaussian mixture model (CGM). The PDF of CGM is given by

$$
f(x) = (1-\varepsilon) \cdot \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left(-\frac{x^2}{2\sigma_g^2}\right) + \frac{\varepsilon\gamma}{\pi(x^2 + \gamma^2)} \tag{7}
$$

where ε is the mixture ratio, σ_g^2 is the Gaussian variance and $\gamma = \sigma^{\alpha}$ is the dispersion of $S\alpha S$ distribution. Such mixture model was first proposed in [9]. Using EM algorithm, Swami achieved the parameters estimation of such model [7]. However, his approach is more expensive computationally due to iterative estimation for triple parameters $(\varepsilon, \sigma, \gamma)$.

To achieve tractable approximation and less computational burden, we consider a Simplified Cauchy Gaussian Mixture model (SCGM) with bi-parameter is defined as

$$\begin{aligned} f(x) &= (1-\varepsilon) f_G(x) + \varepsilon f_C(x) \\ &= (1-\varepsilon) \cdot \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{x^2}{4\sigma^2}\right) + \frac{\varepsilon\sigma}{\pi(x^2+\sigma^2)} \end{aligned} \quad (8)$$

where ε is the mixture ratio, σ is the scale exponent of $S\alpha S$ distribution. We calculate the ratio parameter ε by the following method that utilizes the Fractional Lower Order Moment (FLOM) of $S\alpha S$ random samples,

$$E(|x|^p) = \int_{-\infty}^{\infty} |x|^p [(1-\varepsilon) f_G(x) + \varepsilon f_C(x)] dx \quad (9)$$

where $p < \alpha$. Meanwhile, we know that

$$E(|x|^p) = C(p, \alpha) \cdot \sigma^p \quad (10)$$

where

$$C(p, \alpha) = \frac{2^{p+1} \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\sqrt{\pi}\Gamma\left(-\frac{p}{2}\right)}.$$

For Cauchy and Gaussian distribution, let

$$m_C^p = \int_{-\infty}^{\infty} |x|^p f_C(x) dx = C(p,1) \cdot \sigma^p = \frac{2^{p+1} \Gamma\left(\frac{p+1}{2}\right) \Gamma(-p)}{\sqrt{\pi}\Gamma\left(-\frac{p}{2}\right)} \cdot \sigma^p \quad \text{and}$$

$$m_G^p = \int_{-\infty}^{\infty} |x|^p f_G(x) dx = C(p,2) \cdot (\sqrt{2}\sigma)^p = \frac{2^p \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \cdot (\sqrt{2}\sigma)^p.$$

So, the equation (9) can be rewritten as $E(|x|^p) = (1-\varepsilon) m_G^p + \varepsilon m_C^p$. It is easy to see that

$$\varepsilon = \frac{E\left(|x|^p\right) - m_G^p}{m_C^p - m_G^p}. \quad (11)$$

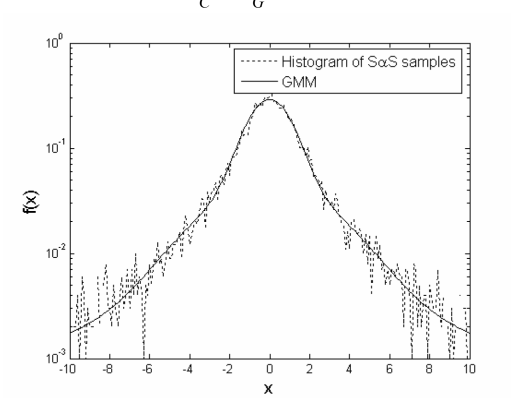

Fig.2 Comparisons between observed histogram and GMM

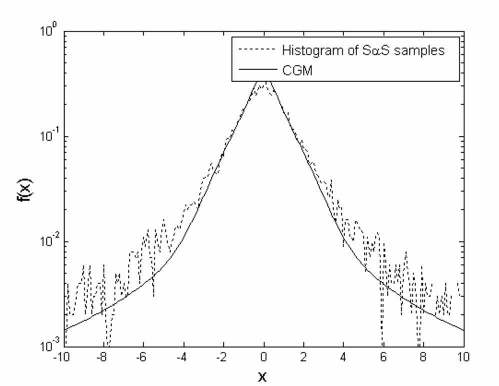

Fig.3 Comparisons between observed histogram and CGM

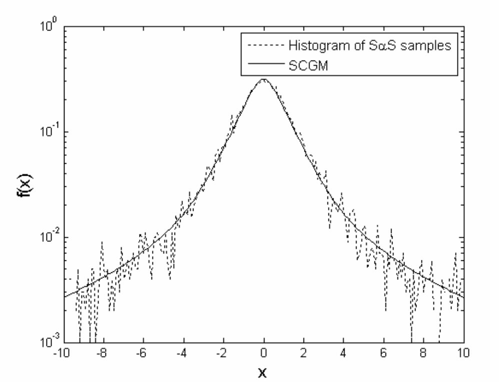

Fig.4 Comparisons between observed histogram and SCGM

In addition, parameter α and σ can be evaluated by Empirical Character Function (ECF) approach. To access the approximation performance of SCGM, we generate 10000 samples subject to $S_\alpha(\sigma, 0, 0)$ with $\sigma = 1$. The comparisons between observed histogram and GMM, CGM, SCGM are shown in Fig.2~Fig.4 respectively. Fig.3 and Fig.4 show that SCGM achieves un-inferior approximation than CGM. Meanwhile SCGM has less computational burden without EM algorithm.

4. Locally optimum detector

4.1. Problem formulation

Detection of deterministic direct current signal $s(i)$ and $A > 0$ in additional noise $n(i)$, $i = 1, 2, \cdots, N$ can be formulated as a hypothesis- testing problem.

$$\begin{aligned} H_1 &: x(i) = As(i) + n(i) \\ H_0 &: x(i) = n(i) \end{aligned} \quad (12)$$

where H_1 is the alternative hypothesis and H_0 is the null hypothesis which indicate the presence and nonpresence of the signal in the observation, respectively. $A(i)$ represents the detected signal.

To decide between the two hypotheses H_0 and H_1, the optimum receiver computes the test statistic that follows from the Neyman-Pearson lemma [2] η

$$\Lambda_{NP}(x)=\sum_i \log\left[\frac{f(x(i)-As(i))}{f(x(i))}\right] \tag{13}$$

and compares it to a threshold η. When $\Lambda_{NP}\geq\eta$, the detector decides that signal occurs in the observation, otherwise, that only noise occurs. In strong interference circumstance, the transmitted signal is quite weak compared with interference. In such case, the Locally Optimum (LO) detectors [6] are most powerful tool. The log-likelihood test for a LO detector is given by

$$\Lambda_{LO}(x)=\sum_{i=1}^{N} As(i)g(x(i)) \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{14}$$

where $g(x)$ is the ZMNL function.

4.2. ZMNL

A standard approach to handing Heavy-tailed noise is to pass it through a zero-memory non-linear (ZMNL) limiter. Consider the SCGM density $f(x)=(1-\varepsilon)f_G(x)+\varepsilon f_C(x)$. Consequently, $f'(x)=(1-\varepsilon)f_G'(x)+\varepsilon f_C'(x)$

$=\dfrac{(\varepsilon-1)x}{4\sigma^3\sqrt{\pi}}\exp\left(-\dfrac{x^2}{4\sigma^2}\right)-\dfrac{\sigma\varepsilon}{\pi}\dfrac{2x}{(x^2+\sigma^2)^2}$. Finally, the ZMNL function $g(x)=-\dfrac{f'(x)}{f(x)}$

$$=\frac{\sqrt{\pi}(x^2+\sigma^2)^2(1-\varepsilon)x\cdot\exp\left(-\dfrac{x^2}{4\sigma^2}\right)+8\sigma^4\varepsilon x}{2\sigma^2\sqrt{\pi}(x^2+\sigma^2)^2(1-\varepsilon)\cdot\exp\left(-\dfrac{x^2}{4\sigma^2}\right)+4\sigma^4\varepsilon(x^2+\sigma^2)}.$$

The ZMNL functions are shown in Fig.5 with respect to $\alpha=1.2,1.6,1.8$ and with $\sigma=1$.

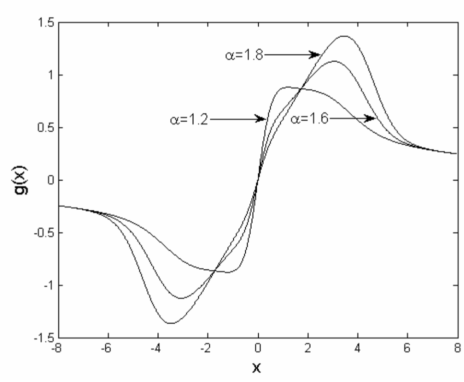

Fig.5 The ZMNL functions with respect to $\alpha=1.2,1.6,1.8$ and with $\sigma=1$.

4.3. The Structure of Adaptive Detector

Based on the above SCGM mixture densities and ZMNL limiter, the adaptive deterministic direct current signal detector that keeps constant false alarm rate (CFAR) can be constructed as shown in Fig.6. For input observation, its parameters α and σ are evaluated by ECF method and then the mixture ratio ε is obtained by equation (11). With certain false alarm ratio (FAR), the corresponding threshold value is calculated via SCGM with ε and σ. Due to the threshold η concerning with Fisher information that is a function with noise parameters α and σ, the presented detector is adaptive and keeps CFAR.

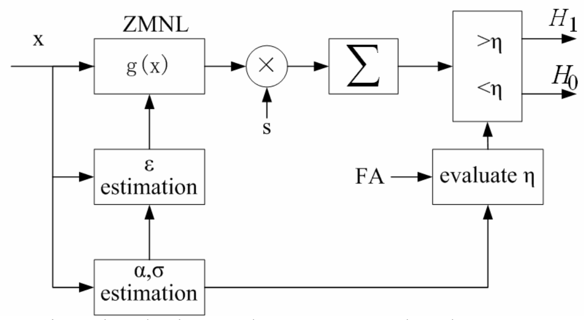

Fig.6 The adaptive LO detector structure based on SCGM

5. Experimental results

Using Monte Carlo simulation, we evaluated the detection probability of three detectors, the optimal, SCGM based and Cauchy detector. The results are based on 10000 realizations of x that is of vector size 20. The corresponding receiver operating characteristics (ROCs) shown in Fig.7 and Fig.8 demonstrate the performance of the detectors in $S\alpha S$ noise environment with $\alpha=1.5$ and $\sigma=1$ with respect to $A=0.5$ and $A=0.2$ versus P_{FA} ranging from 0.001 to 0.45.

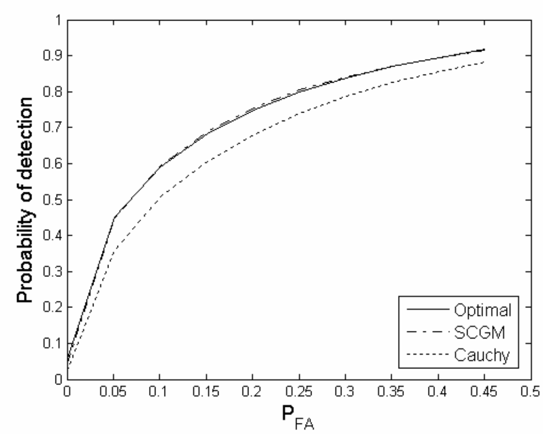

Fig.7 ROCs for optimal, SCGM and Cauchy detector with A=0.5

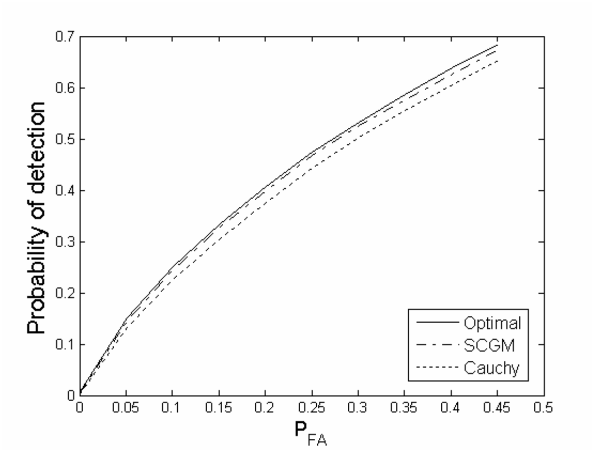

Fig.8 ROCs for optimal, SCGM and Cauchy detector with A=0.2

From the results, it is easy to see that the SCGM based detector has better performance than Cauchy detector and achieved near optimal detection performance.

6. Conclusion

In this paper, we introduced a new non-Gaussian mixture model with bi-parameter for approximation $S\alpha S$ interference density. Then, we presented the approach to evaluate mixture ratio parameter. Based on such model, we deduced the ZMNL function for LO detection. Finally, we constructed the adaptive deterministic direct current signal detector that keeps CFAR. The simulation results demonstrate the detector has near optimal performance and superiority to Cauchy detector.

Acknowledgments

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